New Results on Permutation Pattern-Replacement with a Generalization of Erdős-Szekeres

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***Project summary**: this project used combinatorics to give new insights into permutation patterns, which have many potential applications in computer science as well as in automatic code optimization theory.*

Abstract

In this paper we study pattern-replacement equivalence relations on the set S_n of permutations of length n. Each equivalence relation is determined by a set of patterns, and equivalent permutations are connected by pattern-replacements in a manner similar to that of the Knuth relation.

One of our main results generalizes the celebrated Erdos-Szekeres Theorem for permutation pattern-avoidance to a new result for permutation pattern-replacement. In particular, we show that under the $\{123 \cdots k, k \cdots 321\}$-equivalence, all permutations in S_n are equivalent up to parity when $n \geq \Omega(k^2)$.

Additionally, we extend the work of Kuszmaul and Zhou on an infinite family of pattern-replacement equivalences known as the rotational equivalences. Kuszmaul and Zhou proved that the rotational equivalences always yield either one or two nontrivial equivalence classes in S_n, and conjectured that the number of nontrivial classes depended only on the patterns involved in the rotational equivalence (rather than on n). We present a counterexample to their conjecture, and prove a new theorem fully classifying (for large n) when there is one nontrivial equivalence class and when there are two nontrivial equivalence classes.

Finally, we computationally analyze the pattern-replacement equivalences given by sets of pairs of patterns of length four. We then focus on three cases, in which the number of nontrivial equivalence classes matches an OEIS sequence. For two of these we present full proofs of the enumeration and for the final we suggest a potential future method of proof.